

A coupled Volterra system and its exact solutions

S. Y. Lou^{1,2}, Bin Tong¹, Man Jia² and Jin-hua Li²

¹*Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, China*

²*Department of Physics, Ningbo University, Ningbo, 315211, China*

Abstract

A coupled Volterra system is proposed. The model can be considered as one of the integrable discrete form of the coupled integrable KdV system which is a significant physical model. Many types of cnoidal waves, positons, negatons (solitons) and complexitons of the model are obtained by a simple rational expansion method of the Jacobi elliptic functions, trigonometric functions and hyperbolic functions.

I. INTRODUCTION.

The Volterra system [1]

$$a_{nt} - a_n(a_{n-1} - a_{n+1}) = 0, \quad (1)$$

is one of the famous integrable differential-difference systems which has been applied in various physical systems such as the network, statistical physics and biology [2]. Some types of exact solutions of the model have been studied by many authors (say [3]).

It is also interesting that the Volterra system is one of the simplest discrete form of the KdV equation. In fact, if we write

$$a_n = 1 + \delta^2 u \left((n - 2t)\delta, \frac{1}{3}\delta^3 t \right) + O(\delta^3) \quad (2)$$

$$\equiv 1 + \delta^2 u(x, \tau) + O(\delta^3), \quad (3)$$

$$a_{n\pm 1} = 1 + \delta^2 u(x \pm \delta, \tau) + O(\delta^3), \quad (4)$$

then (1) becomes the well known KdV equation

$$\frac{1}{3}(u_\tau + 6uu_x + u_{xxx})\delta^5 + O(\delta^6) = 0. \quad (5)$$

Recently, some types of integrable coupled KdV system have been derived from some different physical fields including the atmospheric dynamics [4], Bose-Einstein condensation [5] and two-wave modes in a shallow stratified liquid [6]. A common special interesting case of [4]–[6] has the following form

$$u_t + 6\alpha vv_x + 6uu_x + u_{xxx} = 0, \quad (6)$$

$$v_t + 6vu_x + 6uv_x + v_{xxx} = 0. \quad (7)$$

Some kinds of analytical negatons, positons and complexitons of the coupled KdV system (6)–(7) for $\alpha = -1$ are studied in [7].

On the other hand, with the development of the computer science and the discreteness of the micro physics, to look for the integrable discrete forms of the useful continuous integrable models becomes a hot topic in nonlinear science. Actually, one continuous integrable model may have some different integrable discrete forms. For instance, the integrable discrete versions of the KdV equation may be the Volterra equation, the Toda lattice [8], the Ablowitz model [9] and the hybrid lattice [10], etc.

In section II, we propose a coupled Volterra system which is a discrete version of the coupled KdV system (6)–(7) and study the integrability of the coupled Volterra system. The periodic cnoidal waves, solitons (negatons), positons and complexitons are investigated by using some suitable solution ansatzs in section III and section IV for $\alpha > 0$ and $\alpha < 0$, respectively. A short summary is presented in the last section.

II. A COUPLED VOLTERRA SYSTEM

It is interesting that the following coupled Volterra system

$$a_{nt} - a_n(a_{n-1} - a_{n+1}) - \alpha b_n(b_{n-1} - b_{n+1}) = 0, \quad (8)$$

$$b_{nt} - a_n(b_{n-1} - b_{n+1}) - b_n(a_{n-1} - a_{n+1}) = 0, \quad (9)$$

is an integrable extension of the usual Volterra system (1). It is obvious that both $b_n = 0$ and $b_n = \frac{1}{\sqrt{\alpha}}a(n, t)$ reduce the coupled Volterra system (8)–(9) to the usual Volterra equation (1).

The integrability of the coupled Volterra system (8)–(9) is guaranteed by the following theorem.

Theorem. *The coupled Volterra system (8)–(9) possesses the following Lax pair,*

$$L\psi_n = \Lambda\psi_n, \quad (10)$$

$$\psi_{nt} = M\psi_n, \quad (11)$$

where

$$L \equiv \begin{pmatrix} a_n T_+ + T_- & \alpha b_n T_+ \\ b_n T_+ & a_n T_+ + T_- \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & \alpha \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}, \quad \psi_n \equiv \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}, \quad (12)$$

$$M \equiv - \begin{pmatrix} a_n a_{n+1} + \alpha b_n b_{n+1} & \alpha(a_n b_{n+1} + b_n a_{n+1}) \\ a_n b_{n+1} + b_n a_{n+1} & a_n a_{n+1} + \alpha b_n b_{n+1} \end{pmatrix} T_+^2, \quad (13)$$

and T_+ and T_- are shift operators defined by

$$T_+ f_n \equiv f_{n+1}, \quad T_- f_n \equiv f_{n-1} \quad (14)$$

for an arbitrary function f_n .

Proof. By the direct calculations, one can prove that the matrices M and Λ are commutable

and then the compatibility condition of (12) and (13) is just the zero curvature condition

$$L_t + LM - ML = 0. \quad (15)$$

Substituting the definition equations of L and M into (15) just leads to the coupled Volterra system (8)–(9). The theorem is proved.

Another interesting fact is that the coupled Volterra system (8)–(9) is really a discrete form of the coupled KdV system (6)–(7). Applying the following continuous limiting procedure

$$\begin{aligned} a_n &= 1 + \delta^2 u \left((n - 2t)\delta, \frac{1}{3}\delta^3 t \right) + O(\delta^3) \\ &\equiv 1 + \delta^2 u(x, \tau) + O(\delta^3), \end{aligned} \quad (16)$$

$$\begin{aligned} b_n &= \delta^2 v \left((n - 2t)\delta, \frac{1}{3}\delta^3 t \right) + O(\delta^3) \\ &\equiv \delta^2 v(x, \tau) + O(\delta^3), \end{aligned} \quad (17)$$

$$a_{n\pm 1} = 1 + \delta^2 u(x \pm \delta, \tau) + O(\delta^3), \quad (18)$$

$$b_{n\pm 1} = \delta^2 v(x \pm \delta, \tau) + O(\delta^3) \quad (19)$$

to (8)–(9), we have

$$\frac{1}{3}(u_\tau + 6\alpha vv_x + 6uu_x + u_{xxx})\delta^5 + O(\delta^6) = 0, \quad (20)$$

$$\frac{1}{3}(v_\tau + 6vu_x + 6uv_x + v_{xxx})\delta^5 + O(\delta^6) = 0 \quad (21)$$

which is just the special coupled KdV system (6)–(7) with $t \rightarrow \tau$.

III. CNOIDAL WAVES, SOLITONS AND POSITONS OF THE COUPLED VOLTERRA SYSTEM FOR $\alpha > 0$

In the continuous case, the rational expansion of the elliptic and hyperbolic functions is one of the simplest methods to find travelling periodic and solitary wave solutions. Fortunately, this method is valid also in discrete cases.

To find some types of cnoidal wave solutions of the coupled Volterra system (8)–(9), one may take some types of the elliptic function expansion ansatzs, say,

$$a(n, t) = \frac{P(f, g, h)}{R(f, g, h)}, \quad (22)$$

$$b(n, t) = \frac{Q(f, g, h)}{R(f, g, h)} \quad (23)$$

where $\{P(f, g, h), Q(f, g, h), R(f, g, h)\}$ are polynomial functions of $\{f, g, h\}$ and $f \equiv \text{sn}(kn + ct, m)$, $g \equiv \text{cn}(kn + ct, m)$ and $h \equiv \text{dn}(kn + ct, m)$ are Jacobi elliptic functions with three constant parameters, the wave number k , the angular frequency ω and the modulus m . Because of the computational difficulty, here, we just give some special examples of (22)–(23).

Case 1. The first simple expansion ansatz reads

$$a(n, t) = \frac{a_1 + c_1 \text{cn}^2(kn + ct, m)}{a_2 + c_2 \text{cn}^2(kn + ct, m)}, \quad (24)$$

$$b(n, t) = \frac{A[1 + a_0 \text{cn}^2(kn + ct, m)]}{a_2 + c_2 \text{cn}^2(kn + ct, m)} \quad (25)$$

where $a_1, c_1, a_2, c_2, A, k, m$ and c are constants that should be determined later.

Substituting (24)–(25) into (8)–(9) and vanishing the coefficients of the different powers of the Jacobi elliptic function $\text{cn}^2(kn + ct)$, we can obtain a complicated determining equation system for the undetermined constants. Fortunately, there exist a unique general solution for the determining equation system. The result reads ($s \equiv \text{sn}(k\delta, m)$, $d \equiv \text{dn}(k\delta, m)$, $C \equiv \text{cn}(k\delta, m)$)

$$s^2 = \frac{4a_2c_2(a_2 + c_2)[m^2(a_2 + c_2) - c_2]}{[m^2(a_2 + c_2)^2 - c_2^2]^2}, \quad (26)$$

$$A^2 = \frac{a_2c_2c^2(a_2 + c_2)c^2[m^2(a_2 + c_2) - c_2]}{\alpha(a_2a_0 - c_2)^2}, \quad (27)$$

$$a_1 = \frac{cm^2s(a_2 + c_2)(a_2 - c_2 + 2a_0a_2)[(a_2 + c_2)m^2s^2 - 2c_2]}{4dCc_2(a_0a_2 - c_2)} - \frac{ca_2}{2dCs} \\ + \frac{cc_2s(2a_0a_2 - c_2)(2a_2 + 2c_2 - c_2s^2)}{4dC(a_2 + c_2)(a_0a_2 - c_2)}, \quad (28)$$

$$c_1 = \frac{a_1(a_0c_2 + 2c_2 - a_0a_2)}{a_2 + 2a_0a_2 - c_2} + \frac{c(a_2 + c_2 - c_2s^2)^2(c_2 - a_0a_2)}{2dCs(a_2 + c_2)(a_2 + 2a_0a_2 - c_2)}, \quad (29)$$

while the constants a_2, c_2, c, a_0 and m remain to be free parameters.

From (26) and (27), it is known that the real condition of the periodic solution requires

$$\alpha > 0.$$

Fig. 1 shows the structure of the periodic solution (24) with the parameter selections

$$\alpha = c = a_2 = c_2 = 1, a_0 = 2, \delta = 1, m = 0.999 \quad (30)$$

at time $t = 0$.

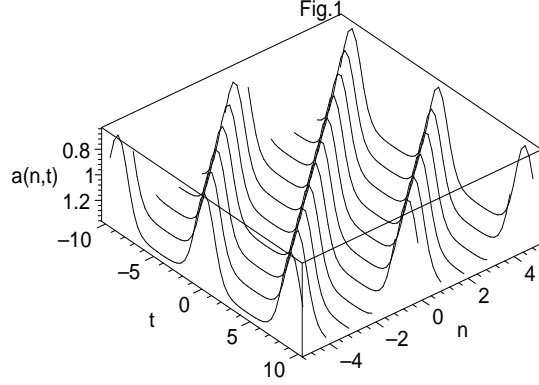


FIG. 1: The structure of the periodic wave expressed by (24) with the parameters (30) at time $t = 0$.

It is remarkable that for the $\alpha > 0$ case, in addition to the above cnoidal wave solution, we can find many other nonequivalent periodic waves by similar procedures. In the following, we just list some of them.

Case 2.

$$a(n, t) = \frac{a_1 + c_1 \text{cn}(kn + ct, m)}{a_2 + c_2 \text{cn}(kn + ct, m)}, \quad (31)$$

$$b(n, t) = \frac{A_1 + A_2 \text{cn}(kn + ct, m)}{a_2 + c_2 \text{cn}(kn + ct, m)} \quad (32)$$

where the constants $a_1, c_1, a_2, c_2, A_1, A_2, k, c$ and m satisfy the following conditions:

$$a_1 = \frac{sc}{4a_2d}(m^2a_2^4 + 1 - m^2), \quad (33)$$

$$\frac{A_2}{A_1} = \frac{2a_2[c + 2c_1sd - cm^2s^2(1 - a_2^2)]}{cs^2[m^2(1 - a_2^2)^2 - 1] + 2a_2^2(c + 2c_1sd)}, \quad (34)$$

$$A_1 = \pm \frac{cs^2[m^2(1 - a_2^2)^2 - 1] + 2a_2^2(c + 2c_1sd)}{4sda_2\sqrt{\alpha}} \quad (35)$$

$$C = 1 + \frac{s^2}{2a_2^2} [m^2(1 - a_2^2)^2 - 1], \quad c_2 = 1. \quad (36)$$

Case 3.

$$a(n, t) = \frac{a_1 + c_1 \text{dn}(kn + ct, m)}{a_2 + c_2 \text{dn}(kn + ct, m)}, \quad (37)$$

$$b(n, t) = \frac{A_1 + A_2 \text{dn}(kn + ct, m)}{a_2 + c_2 \text{dn}(kn + ct, m)} \quad (38)$$

with the constant constraints

$$a_1 = \frac{sc}{4a_2C}(m^2 + a_2^4 - 1) + c_1a_2, \quad (39)$$

$$\frac{A_2}{A_1} = \frac{2a_2[c + 2c_1sC - cs^2(1 - a_2^2)]}{cs^2[(1 - a_2^2)^2 - m^2] + 2a_2^2(c + 2c_1sC)}, \quad (40)$$

$$A_1 = \pm \frac{cs^2[(1 - a_2^2)^2 - m^2] + 2a_2^2(c + 2c_1sC)}{4sCa_2\sqrt{\alpha}} \quad (41)$$

$$d = 1 + \frac{s^2}{2a_2^2} [(1 - a_2^2)^2 - m^2], \quad c_2 = 1. \quad (42)$$

Case 4.

$$a(n, t) = \frac{a_1 + c_1 \text{sn}(kn + ct, m)}{a_2 + c_2 \text{sn}(kn + ct, m)}, \quad (43)$$

$$b(n, t) = \frac{A_1 + A_2 \text{sn}(kn + ct, m)}{a_2 + c_2 \text{sn}(kn + ct, m)} \quad (44)$$

with the constant constraints

$$c_1 = \frac{a_1}{a_2} - \frac{c(Cd - 1)}{2s(1 + m^2a_2^4)}(m^2a_2^4 - 1), \quad (45)$$

$$\frac{A_2}{A_1} = \frac{(1 + m^2a_2^4)(2sa_1 + cdCa_2)}{a_2[m^2a_2^4(ca_2 + 2sa_1) + 2sa_1 + 2cdCa_2 - ca_2]}, \quad (46)$$

$$A_1 = \pm \frac{m^2a_2^4(ca_2 + 2sa_1) + 2sa_1 + 2cdCa_2 - ca_2}{2a_2\sqrt{2\alpha(1 - dC)}(1 + a_2^4m^2)} \quad (47)$$

$$s^2 = \frac{2a_2^2(1 - dC)}{1 + m^2a_2^4}, \quad c_2 = 1. \quad (48)$$

Case 5.

$$a(n, t) = \frac{a_1 \text{cn}(kn + ct, m) + c_1 \text{sn}(kn + ct, m)}{a_2 \text{cn}(kn + ct, m) + c_2 \text{sn}(kn + ct, m)}, \quad (49)$$

$$b(n, t) = \frac{A_1 \text{cn}(kn + ct, m) + A_2 \text{sn}(kn + ct, m)}{a_2 \text{cn}(kn + ct, m) + c_2 \text{sn}(kn + ct, m)} \quad (50)$$

with the constant constraints

$$c_1 = \frac{a_1}{a_2} - \frac{c(d - 1)(m^2a_2^4 - a_2^4 + 1)}{2sC[m^2a_2^4 - (1 + a_2^2)^2]}, \quad (51)$$

$$\frac{A_2}{A_1} = \frac{[(1 + a_2^2)^2 - m^2a_2^4](cda_2 + 2sCa_1)}{a_2[m^2a_2^4(ca_2 + 2sa_1C) - (1 + a_2^2)(2sCa_1 + 2cda_2 - ca_2 + ca_2^3 + 2sCa_1a_2^2)]}, \quad (52)$$

$$A_1 = \pm \frac{m^2a_2^4(ca_2 + 2sa_1C) - (1 + a_2^2)(2sCa_1 + 2cda_2 - ca_2 + ca_2^3 + 2sCa_1a_2^2)}{4a_2C(1 - d)\sqrt{\alpha}} \quad (53)$$

$$s^2 = \frac{2a_2^2(1 - d)}{(1 + a_2^2)^2 - m^2a_2^4}, \quad c_2 = 1. \quad (54)$$

Case 6.

$$a(n, t) = \frac{a_1 \operatorname{dn}(kn + ct, m) + c_1 \operatorname{sn}(kn + ct, m)}{a_2 \operatorname{dn}(kn + ct, m) + c_2 \operatorname{sn}(kn + ct, m)}, \quad (55)$$

$$b(n, t) = \frac{A_1 \operatorname{dn}(kn + ct, m) + A_2 \operatorname{sn}(kn + ct, m)}{a_2 \operatorname{dn}(kn + ct, m) + c_2 \operatorname{sn}(kn + ct, m)} \quad (56)$$

with the constant constraints

$$c_1 = \frac{a_1}{a_2} - \frac{c(C-1)(m^4 a_2^4 - 1 - m^2 a_2^4)}{2sd[(1 + m^2 a_2^2)^2 - m^2 a_2^4]}, \quad (57)$$

$$\frac{A_2}{A_1} = \frac{[(1 + m^2 a_2^2)^2 - m^2 a_2^4](cCa_2 + 2sda_1)}{a_2[2sda_1[(1 + m^2 a_2^2)^2 - m^2 a_2^4] + ca_2[m^4 a_2^4 + m^2 a_2^2(2C - a_2^2) + 2C - 1]]}, \quad (58)$$

$$A_1 = \pm \frac{s[2sda_1[(1 + m^2 a_2^2)^2 - m^2 a_2^4] + ca_2[m^4 a_2^4 + m^2 a_2^2(2C - a_2^2) + 2C - 1]]}{4a_2^2 d(1 - C)\sqrt{\alpha}}, \quad (59)$$

$$s^2 = \frac{2a_2^2(1 - C)}{(1 + a_2^2 m^2)^2 - m^2 a_2^4}, \quad c_2 = 1. \quad (60)$$

Case 7.

$$a(n, t) = \frac{a_1 \operatorname{dn}(kn + ct, m) + c_1 \operatorname{cn}(kn + ct, m)}{a_2 \operatorname{dn}(kn + ct, m) + c_2 \operatorname{cn}(kn + ct, m)}, \quad (61)$$

$$b(n, t) = \frac{A_1 \operatorname{dn}(kn + ct, m) + A_2 \operatorname{cn}(kn + ct, m)}{a_2 \operatorname{dn}(kn + ct, m) + c_2 \operatorname{cn}(kn + ct, m)} \quad (62)$$

with the constant constraints

$$c_1 = \frac{a_1}{a_2} - \frac{c(Cd - 1)(m^2 a_2^4 - 1)}{2s(1 + m^2 a_2^4)}, \quad (63)$$

$$\frac{A_2}{A_1} = \frac{(1 + m^2 a_2^4)(cCda_2 + 2sa_1)}{a_2[m^2 a_2^4(ca_2 + 2sa_1) + 2sa_1 + 2cda_2 C - ca_2]}, \quad (64)$$

$$A_1 = \pm \frac{m^2 a_2^4(ca_2 + 2sa_1) + 2sa_1 + 2cda_2 C - ca_2}{2a_2 \sqrt{2\alpha(1 - dC)(1 + m^2 a_2^4)}}, \quad (65)$$

$$s^2 = \frac{2a_2^2(1 - dC)}{1 + a_2^4 m^2}, \quad c_2 = 1. \quad (66)$$

Especially, when $m \rightarrow 1$, the previous periodic wave solutions become negaton (soliton) solutions. For instance, (24) and (25) become the soliton solution

$$a(n, t) = \frac{a_1 + c_1 \operatorname{sech}^2(kn + ct)}{a_2 + c_2 \operatorname{sech}^2(kn + ct)}, \quad (67)$$

$$b(n, t) = \frac{A[1 + a_0 \operatorname{sech}^2(kn + ct)]}{a_2 + c_2 \operatorname{sech}^2(kn + ct)} \quad (68)$$

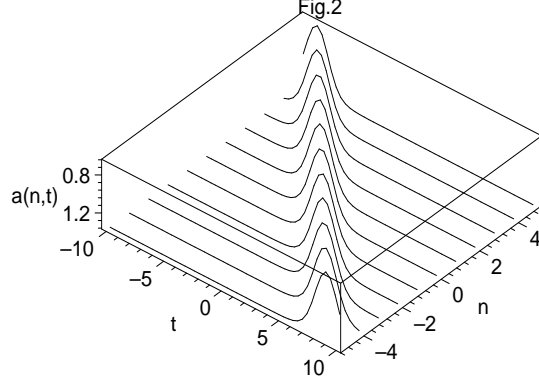


FIG. 2: The structure of the solitary wave expressed by (67) which is the limit case of the figure 1 for $m = 1$.

with

$$S^2 = \frac{4c_2(a_2 + c_2)}{(a_2 + 2c_2)^2}, \quad A_1^2 = \frac{c_2 a_2^2 c^2 (a_2 + c_2)}{\alpha (a_2 a_0 - c_2)^2}, \quad (69)$$

$$a_1 = \frac{2ca_2c_2(a_2 + c_2)}{S(a_2a_0 - c_2)(a_2 + 2c_2)} - \frac{a_2^3c}{2S(a_2a_0 - c_2)^2}, \quad (70)$$

$$c_1 = \frac{2cc_2^2(a_2 + c_2)}{S(a_2a_0 - c_2)(a_2 + 2c_2)} - \frac{c_2c(5a_2^2 + 12a_2c_2 + 8c_2^2)}{2S(a_2a_0 - c_2)^2}, \quad (71)$$

Fig. 2 shows the structure of the soliton solution (24) with the same parameter selections as (30) except for $m = 1$.

In [7], for the $\alpha < 0$ case, it is found that the coupled KdV system (6) and (7) possesses not only the analytic solitons (negatons) but also analytical positons and complexitons. Now the natural question is whether the coupled Volterra system possesses analytical positons and/or complexitons.

Similarly, all the cnoidal wave solutions presented above will reduced to the positon solutions when we take $m = 0$. For instance, the first type of the positon solutions can be obtained from the periodic solution (24) and (25) by taking $m = 0$, which has the form of

$$a(n, t) = \frac{2[a_1 + c_1 \cos^2(kn + ct)]}{\cos(2kn + 2ct) \pm \cos(k\delta)}, \quad (72)$$

$$b(n, t) = \frac{2A_1[1 + a_0 \cos^2(kn + ct)]}{\sqrt{\alpha}[\cos(2kn + 2ct) \pm \cos(k\delta)]}, \quad (73)$$

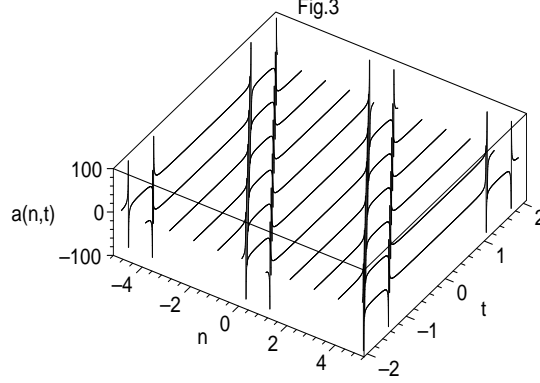


FIG. 3: A typical positon structure expressed by (72)–(76) with the parameter selections (77).

where

$$a_1 = \frac{c [4 + a_0(2 + \cos(2k\delta) + \cos(4k\delta)) \mp (1 + a_0) \cos(k\delta) \cos(2k\delta)]}{4 \sin(2k\delta)[2 + a_0 \mp a_0 \cos(k\delta)]}, \quad (74)$$

$$c_1 = -\frac{c[2 + a_0 \mp a_0 \cos(k\delta) \cos(2k\delta)]}{\sin(2k\delta)[2 + a_0 \mp a_0 \cos(k\delta)]}, \quad (75)$$

$$A_1^2 = \frac{\sin(k\delta)^2}{[2 + a_0 \mp a_0 \cos(k\delta)]^2}, \quad (76)$$

and c , a_0 and k are arbitrary constants.

Obviously, the positon solution (72)–(73) is always singular. Fig. 3 shows a special structure of (72) and (73) with the parameter selections

$$c = \alpha = \delta = 1, \quad a_0 = 2, \quad k = 0.6435011088. \quad (77)$$

Actually, all the positon solutions obtained from Cases 1 to 7 by taking $m = 0$ are singular except for the trivial constant solution in Case 3.

For a coupled nonlinear system, there may be different types of solitons and positons. The first type of soliton solutions given by (67) and (68) possesses the property that the fields $a(n, t)$ and $b(n, t)$ both have the ring or bell shape. The coupled Volterra system can have other kinds of soliton solutions. For instance, by substituting the following solution ansatz

$$a(n, t) = \frac{a_0 + a_1 \cosh(kn + ct) + a_2 \cosh(2kn + 2ct)}{b_0 + b_1 \cosh(kn + ct) + b_2 \cosh(2kn + 2ct)}, \quad (78)$$

$$b(n, t) = \frac{d_0 \sinh(kn + ct)}{b_0 + b_1 \cosh(kn + ct) + b_2 \cosh(2kn + 2ct)} \quad (79)$$

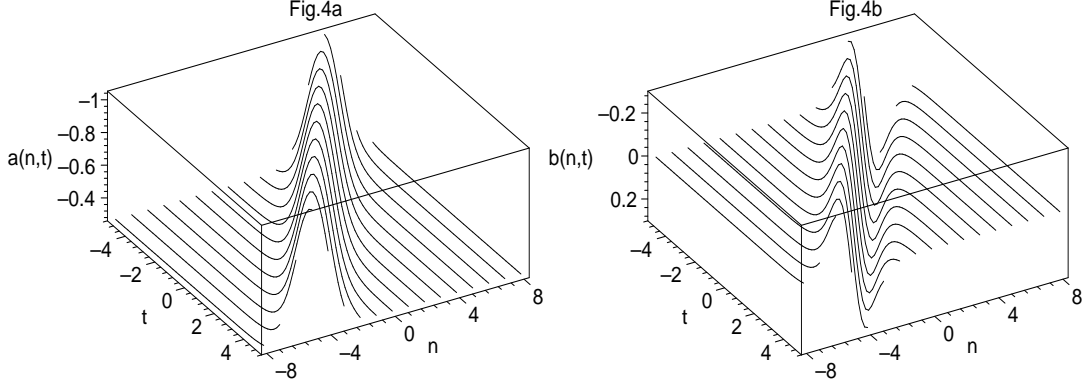


FIG. 4: Second type of soliton structure expressed by (78)–(85) with the parameter selections (86).

into the coupled Volterra system (8) and (9), one can find that

$$d_0 = \mp c \sin(c_0) \left[\cos\left(\frac{3}{2}k\delta\right) - \cos\left(\frac{1}{2}k\delta\right) \right], \quad (80)$$

$$b_0 = \sqrt{\alpha} \sin(k\delta)(1 + \cos(2c_0) + \cos(k\delta)), \quad (81)$$

$$b_1 = \pm 4\sqrt{\alpha} \sin(k\delta) \cos(c_0) \cos\left(\frac{1}{2}k\delta\right), \quad (82)$$

$$a_0 = -\frac{c}{2} \left[\cos(2c_0) + 2 \cos\left(\frac{1}{2}k\delta\right) \cos\left(\frac{3}{2}k\delta\right) \right], \quad (83)$$

$$a_1 = \mp c \cos(c_0) \left[\cos\left(\frac{1}{2}k\delta\right) + \cos\left(\frac{3}{2}k\delta\right) \right], \quad (84)$$

$$b_2 = \sqrt{\alpha} \sin(k\delta), \quad a_2 = -\frac{c}{2}, \quad (85)$$

where c , c_0 and k are arbitrary constants.

From (80)–(85), we know that this kind of soliton solution is also valid only for $\alpha > 0$. The solution (78)–(79) may be singular or analytical based on the different selections of the parameters.

Fig. 4 exhibits the structure of the soliton solution given by (78)–(85) with the special parameter selections

$$\delta = c_0 = \alpha = 1, \quad c = k = 2, \quad (86)$$

for the upper sign. It is clear that in this case the soliton structures are different for the fields $a(n, t)$ and $b(n, t)$. The soliton structure for $a(n, t)$ is still a bell or ring shape while that for $b(n, t)$ becomes staggered.

It is noted that because of the arbitrariness of the constants k , c and c_0 , if we take

$$k \rightarrow \sqrt{-1}k, \quad c \rightarrow \sqrt{-1}c, \quad c_0 \rightarrow \sqrt{-1}c_0 \quad (87)$$

then the soliton solution (78)–(85) is transformed to another type of positon solutions

$$a(n, t) = \frac{a_0 + a_1 \cos(kn + ct) + a_2 \cos(2kn + 2ct)}{b_0 + b_1 \cos(kn + ct) + b_2 \cos(2kn + 2ct)}, \quad (88)$$

$$b(n, t) = \frac{d_0 \sin(kn + ct)}{b_0 + b_1 \cos(kn + ct) + b_2 \cos(2kn + 2ct)} \quad (89)$$

with

$$d_0 = \mp c \sinh(c_0) \left[\cosh\left(\frac{3}{2}k\delta\right) - \cosh\left(\frac{1}{2}k\delta\right) \right], \quad (90)$$

$$b_0 = \sqrt{\alpha} \sinh(k\delta) [1 + \cosh(2c_0) + \cosh(k\delta)], \quad (91)$$

$$b_1 = \pm 4\sqrt{\alpha} \sinh(k\delta) \cosh(c_0) \cosh\left(\frac{1}{2}k\delta\right), \quad (92)$$

$$a_0 = -\frac{c}{2} \left[\cosh(2c_0) + 2 \cosh\left(\frac{1}{2}k\delta\right) \cosh\left(\frac{3}{2}k\delta\right) \right], \quad (93)$$

$$a_1 = \mp c \cosh(c_0) \left[\cosh\left(\frac{1}{2}k\delta\right) + \cosh\left(\frac{3}{2}k\delta\right) \right], \quad (94)$$

$$b_2 = \sqrt{\alpha} \sinh(k\delta), \quad a_2 = -\frac{c}{2}. \quad (95)$$

Fig. 5 displays the structure of the positon solution (88)–(95) for the upper sign with the special parameter selections

$$\delta = \alpha = 1, \quad c = k = \frac{\pi}{2}, \quad c_0 = \frac{\pi}{3}. \quad (96)$$

Evidently, in this case the positon structure is still singular. Up to now, for the $\alpha > 0$ case, we have not yet found any analytical positons and complexitons. However, similar to the single KdV equation and the Toda system [11], we believe that the coupled Volterra system (8) and (9) for $\alpha > 0$ does not possess analytical positons and complexitons.

IV. SOLITONS, POSITONS AND COMPLEXITONS OF THE COUPLED VOLTERRA SYSTEM FOR $\alpha < 0$

For the coupled real Volterra system (8) and (9) with $\alpha < 0$, all the cnoidal wave solution forms in the last section are not valid, however, the function expansion ansatz (78) and (79) is still applicable to obtain some soliton solutions.

Similar to the last section, after substituting (78) and (79) into (8)–(9) for $\alpha < 0$ and

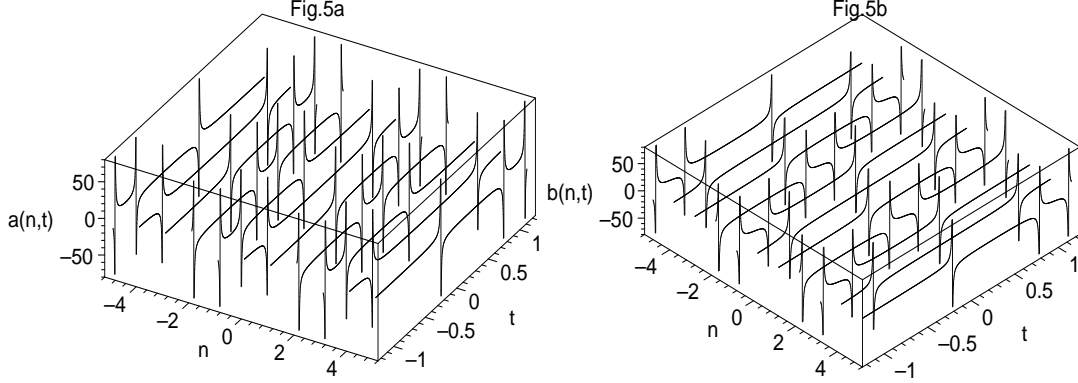


FIG. 5: A typical positon structure expressed by (88)–(95) with the parameter selections (96).

solving the determining equations of the parameters, one can find that

$$d_0 = \frac{2c \sin(c_0) \sinh\left(\frac{1}{2}k\delta\right)}{\sqrt{-\alpha}}, \quad (97)$$

$$b_0 = 1 + \cos(2c_0) + \cosh(k\delta), \quad (98)$$

$$b_1 = 4 \cos(c_0) \cosh\left(\frac{1}{2}k\delta\right), \quad (99)$$

$$a_0 = \frac{c}{\sinh(k\delta)} [\cos(2c_0) + \cosh(k\delta) + \cosh(2k\delta)], \quad (100)$$

$$a_1 = \frac{c \cos(c_0)}{\sinh(k\delta)} \left[\cosh\left(\frac{1}{2}k\delta\right) + \cosh\left(\frac{3}{2}k\delta\right) \right], \quad (101)$$

$$b_2 = 1, \quad a_2 = \frac{c}{2 \sinh(k\delta)}, \quad (102)$$

where c , c_0 and k are arbitrary constants.

Fig. 6 shows the structure of the soliton solution expressed by (78) and (79) with (97)–(102) and the parameter selections

$$c_0 = \frac{\pi}{2}, \quad c = k = \frac{1}{5}, \quad \delta = 1, \quad \alpha = -1. \quad (103)$$

In the continuous case, it has been proved that the coupled KdV system (6) and (7) with $\alpha < 0$ have some types of analytical positon and complexiton solutions [7]. It is interesting that in the discrete case, the coupled Volterra system also possesses analytical positons and complexitons.

To get analytical positon solutions, one can directly apply the constant transformations

$$k \rightarrow \sqrt{-1}k, \quad c \rightarrow \sqrt{-1}c, \quad c_0 \rightarrow \sqrt{-1}c_0,$$

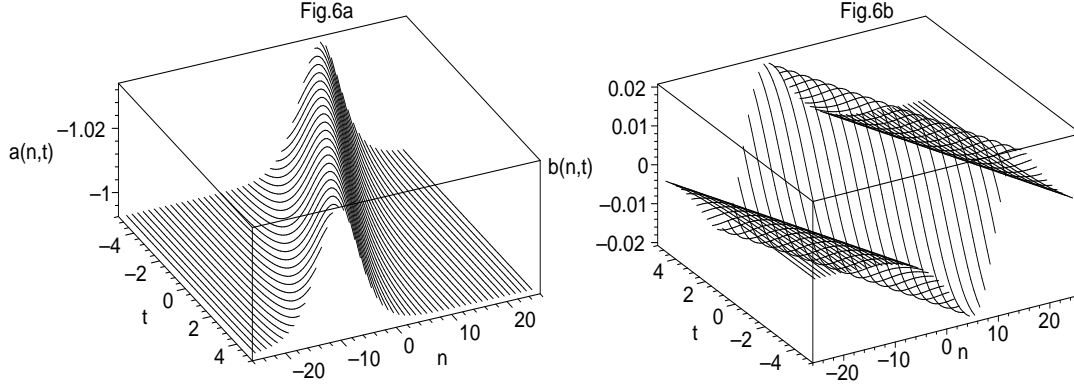


FIG. 6: The structures of the soliton expressed by (a) (78) and (b) (79) with (97)–(102) and the parameter selections (103).

to (78)–(79) with (97)–(102). The result still has the form (88)–(89) but with the constants

$$d_0 = \frac{2c \sinh(c_0) \sin\left(\frac{1}{2}k\delta\right)}{\sqrt{-\alpha}}, \quad (104)$$

$$b_0 = 1 + \cosh(2c_0) + \cos(k\delta), \quad (105)$$

$$b_1 = 4 \cosh(c_0) \cos\left(\frac{1}{2}k\delta\right), \quad (106)$$

$$a_0 = -\frac{c}{2 \sin(k\delta)} [\cosh(2c_0) + \cos(k\delta) + \cos(2k\delta)], \quad (107)$$

$$a_1 = -\frac{c \cosh(c_0)}{\sin(k\delta)} \left[\cos\left(\frac{1}{2}k\delta\right) + \cos\left(\frac{3}{2}k\delta\right) \right], \quad (108)$$

$$b_2 = 1, \quad a_2 = -\frac{c}{2 \sin(k\delta)}, \quad (109)$$

Fig. 7 reveals the structure of the analytical positon solution expressed by (78) and (79) with (104)–(109) and the parameter selections

$$c_0 = 1, \quad c = 1, \quad k = 2, \quad \delta = 1, \quad \alpha = -1. \quad (110)$$

To get analytical complexiton solutions of the coupled Volterra system (8)–(9), we may use the following solution ansatz

$$a(n, t) = \frac{a_0 + a_1 \cosh(\xi_1) \cos(\xi_2) + a_2 \sinh(\xi_1) \sin(\xi_2) + a_3 [\cosh(2\xi_1) + \cos(2\xi_2)]}{b_0 + b_1 \cosh(\xi_1) \cos(\xi_2) + b_2 \sinh(\xi_1) \sin(\xi_2) + b_3 [\cosh(2\xi_1) + \cos(2\xi_2)]}, \quad (111)$$

$$b(n, t) = \frac{d_0 + d_1 \cosh(\xi_1) \cos(\xi_2) + d_2 \sinh(\xi_1) \sin(\xi_2) + d_3 [\cosh(2\xi_2) + \cos(2\xi_1)]}{b_0 + b_1 \cosh(\xi_1) \cos(\xi_2) + b_2 \sinh(\xi_1) \sin(\xi_2) + b_3 [\cosh(2\xi_1) + \cos(2\xi_2)]}, \quad (112)$$

where

$$\xi_i = k_i n + c_i t + \xi_{0i}, \quad i = 1, 2. \quad (113)$$

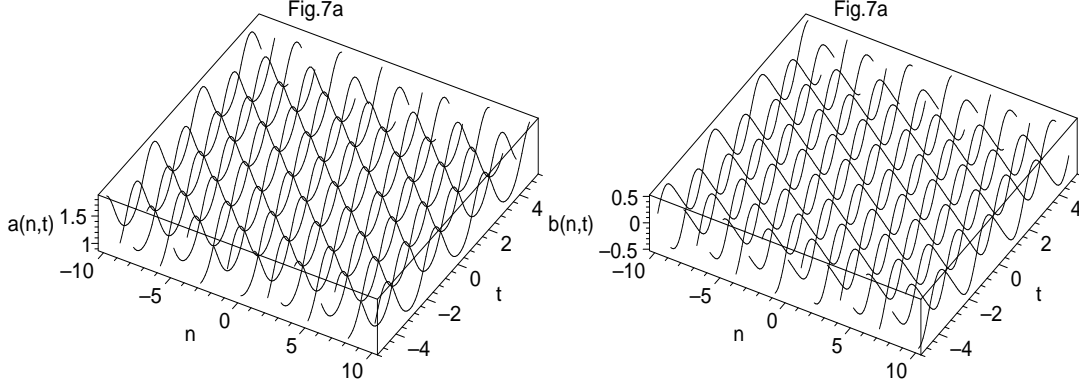


FIG. 7: The structure of the positon expressed by (a) (78) and (b) (79) with (104)–(109) and the parameter selections (110).

After finishing tedious calculations, we find that (8)–(9) really possesses analytical complexiton solutions (an analytical complexiton is just a usual breather) if $\alpha < 0$ while the constants a_j , b_j and c_j for $j = 0, 1, 2$ and 3 should be determined by

$$\begin{aligned} a_0 = & c_1[\cos(4c) \sinh(2b) - \sinh(6b) - \cos(2c) \sinh(4b)] \\ & + c_2[\sin(2c) \cosh(4b) - \sin(4c) \cosh(2b) - \sin(6c)], \end{aligned} \quad (114)$$

$$\begin{aligned} a_1 = & \mp 2 \{ [\sinh(5b) \cos(c) - \cos(5c) \sinh(b) + \cos(c) \sinh(b) + \sinh(3b) \cos(3c)] c_1 \\ & + [\sin(c) \cosh(b) - \sin(c) \cosh(5b) + \sin(5c) \cosh(b) + \sin(3c) \cosh(3b)] c_2 \}, \end{aligned} \quad (115)$$

$$\begin{aligned} a_2 = & \mp 2 \{ [\sin(3c) \cosh(3b) - \sin(5c) \cosh(b) + \sin(c) \cosh(b) + \sin(c) \cosh(5b)] c_1 \\ & + [\sinh(5b) \cos(c) - \sinh(3b) \cos(3c) - \cos(5c) \sinh(b) - \cos(c) \sinh(b)] c_2 \}, \end{aligned} \quad (116)$$

$$a_3 = -2c_2 \cosh(2b) \sin(2c) - 2c_1 \sinh(2b) \cos(2c), \quad (117)$$

$$b_0 = \frac{1}{\sqrt{-\alpha}} [\cos(2c)(1 + 2 \cosh(4b)) + \cosh(2b)(1 - 2 \cos(4c)) + \cosh(6b) - \cos(6c)] \quad (118)$$

$$b_1 = \frac{\mp 4}{\sqrt{-\alpha}} \{ \cosh(b) \cos(5c) + \cosh(b) \cos(3c) - \cos(c) \cosh(5b) - \cos(c) \cosh(3b) \}, \quad (119)$$

$$b_2 = \frac{\pm 4}{\sqrt{-\alpha}} \{ -\sinh(b) \sin(5c) + \sinh(b) \sin(3c) + \sin(c) \sinh(5b) - \sin(c) \sinh(3b) \}, \quad (120)$$

$$b_3 = \frac{2}{\sqrt{-\alpha}} [\cosh(4b) - \cos(4c)], \quad (121)$$

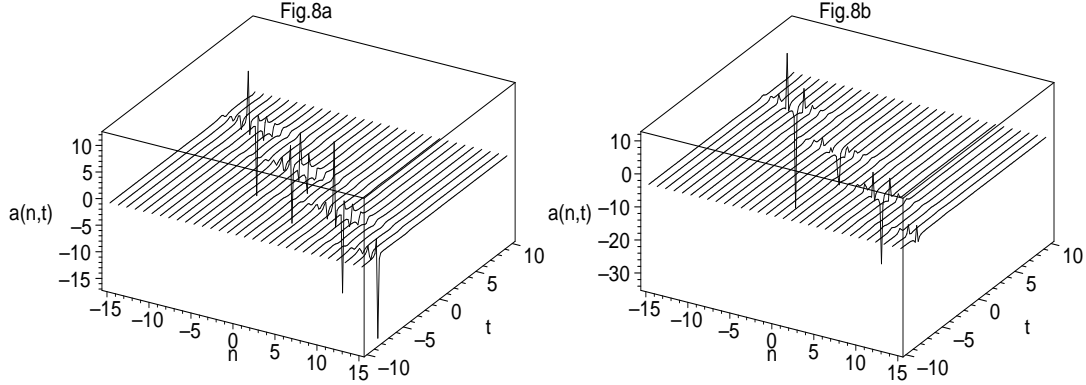


FIG. 8: The structure of the complexiton expressed by (a) (111) and (b) (112) with (114)–(125) and the parameter selections (126).

$$d_0 = -c_2 \cos(4c) \sinh(2b) - c_1 \sin(4c) \cosh(2b) - c_1 \sin(6c) + c_2 \sinh(6b) + c_1 \sin(2c) \cosh(4b) + c_2 \cos(2c) \sinh(4b), \quad (122)$$

$$d_1 = \pm 2 \{ [\sinh(5b) \cos(c) - \cos(5c) \sinh(b) + \cos(c) \sinh(b) + \sinh(3b) \cos(3c)] c_1 + [\sin(c) \cosh(b) - \sin(c) \cosh(5b) + \sin(5c) \cosh(b) + \sin(3c) \cosh(3b)] c_2 \}, \quad (123)$$

$$d_2 = \pm 2 \{ [\cos(5c) \sinh(b) + \cos(c) \sinh(b) - \sinh(5b) \cos(c) + \sinh(3b) \cos(3c)] c_1 + [\sin(3c) \cosh(3b) - \sin(5c) \cosh(b) + \sin(c) \cosh(b) + \sin(c) \cosh(5b)] c_2 \}, \quad (124)$$

$$d_3 = 2c_2 \cos(2c) \sinh(2b) - 2c_1 \sin(2c) \cosh(2b), \quad (125)$$

with

$$b \equiv \frac{1}{2} k_1 \delta, \quad c \equiv \frac{1}{2} k_2 \delta$$

and k_1, k_2, c_1, ξ_{01} and ξ_{02} being arbitrary constants.

Fig. 8 displays the complexiton structure expressed by (111) and (112) with (114)–(125) and the special parameter selections

$$k_2 = \xi_{01} = 0, \quad k_1 = 0.1, \quad c_1 = 0.2, \quad c_2 = 0.7, \quad \xi_{02} = \frac{\pi}{4}, \quad \delta = 1. \quad (126)$$

V. SUMMARY AND DISCUSSIONS

In summary, the special coupled integrable KdV system (6)–(7) is discretized to an integrable coupled Volterra system. The Lax integrability of the coupled Volterra system is proved. By using a simple rational expansion method of the Jacobi elliptic functions, trigonometric functions and hyperbolic functions, various exact solutions are found.

For the coupled Vorterra system (8)–(9) with $\alpha > 0$ there are many types of cnoidal waves described by different types of Jacobi elliptic functions. On one hand, whence the modulus of the cnoidal wave tends to 1, the wave tends to a negaton solution. If a negaton is analytical, then it is called soliton (or solitary wave for nonintegrable systems). It is found that only one of these cnoidal waves can be reduced to a single *analytical* negaton solution when the modulus, m , of the model tends to 1. The soliton solution has a ring or bell shape for both fields $a(n, t)$ and $b(n, t)$. On the other hand, whence the modulus of the cnoidal wave tends to 0, the wave tends to a positon solution. It is found that all the nontrivial positons obtained from the cnoidal waves of section III are singular. This fact is similar to the KdV and Toda system [11].

In the $\alpha > 0$ case, there are two types of single soliton solutions. In addition to the above mentioned analytical negaton solution, there is a different type of solitons which has different shapes for the fields $a(n, t)$ and $b(n, t)$. The new type of soliton solution is obtained by taking a more complicated rational expansion of the hyperbolic functions. Because of the calculation difficulty, we have not yet found any cnoidal wave extension of this type of negaton solution even utilizing computer algebras. The field $a(n, t)$ for the second type of soliton solution also possesses the ring or bell shape while the field $b(n, t)$ possesses a staggered shape.

For the coupled Vorterra system (8)–(9) with $\alpha < 0$, though we have not yet found the cnoidal wave solutions, many other kinds of physically significant solutions, such as the solitons, analytical positons and analytical complexitons are found. The structure of the soliton solution in this case is similar to that of the second type of the solitons for $\alpha > 0$.

It is also interesting that the positon solution can be obtained by many methods. In this paper, we demonstrate that the analytical positon solutions of the coupled Vorterra system (8)–(9) with $\alpha < 0$ can be simply obtained from the negatons by means of the constant analytical extensions.

To obtain analytical complexiton solutions of the coupled Vorterra system (8)–(9) with $\alpha < 0$, a more complicated rational expansion of both the hyperbolic functions and trigonometric functions is used.

Finally it is worth to indicated that the real solutions of the coupled Vorterra system (8)–(9) with $\alpha < 0$ can also be obtained by means of the analytical continuous extensions and vice versa. Some analytical continuous extension examples have been given in section IV. Here,

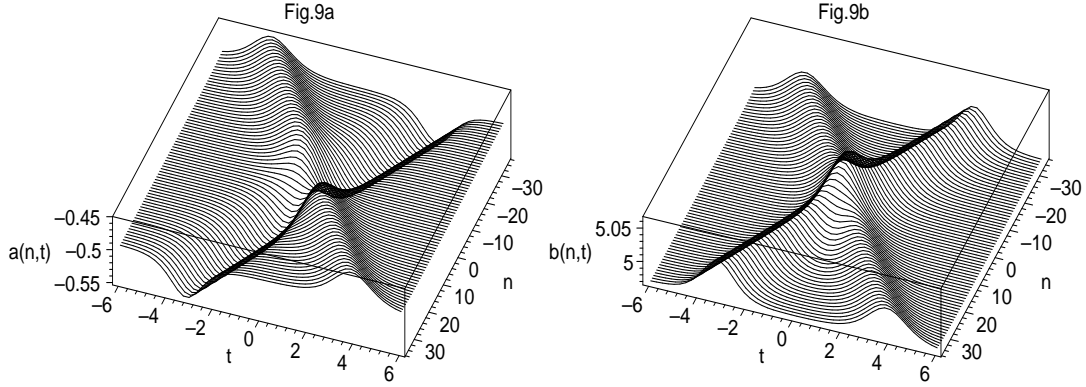


FIG. 9: The evolution of the two-soliton interaction solution expressed by (a) (111) and (b) (112) with the transformations (127) and the parameter selections (128).

we just mention a further interesting example. If we apply the constant transformations

$$k_2 \rightarrow \sqrt{-1}k_2, \quad c_2 \rightarrow \sqrt{-1}c_2, \quad \xi_{02} \rightarrow \sqrt{-1}\xi_{02}, \quad (127)$$

to (111)–(112), then the complexiton solution of $\alpha < 0$ case becomes a two-soliton solution for $\alpha > 0$.

Fig. 9 shows the two-soliton interaction related to (111)–(112) with the transformation (127) under the parameter (after (127)) selections

$$k_2 = \xi_{01} = 0, \quad k_1 = 0.1, \quad c_1 = 0.1, \quad c_2 = 1, \quad \xi_{02} = \frac{\pi}{4}, \quad \delta = 1. \quad (128)$$

Though we have obtained many types of exact solutions of the model via a simple function expansion method, various problems, such as the general multiple soliton solutions and τ function solutions are still open. As a discrete form of the significant physical model, the more about the model will be studied further.

Acknowledgments

The authors are in debt to the helpful discussions with Drs. X. Y. Tang, P. Liu, Y. Gao and X. Y. Jiao. The work was supported by the National Natural Science Foundations of China (Nos. 10475055, 10735030 and 90503006), the Scientific Research Fund of Zhejiang Provincial Education Department (No. 20040969) and National Basic Research Program of

China (973 Program 2007CB814800).

- [1] O. Babelon, Commun. Math. Phys. 266 (2006) 819.
- [2] L. E. Reichl, A Modern Course in Statistical Physics, Edward Arnold (publishers) pp641-644 (1980);
P. A. Damianou and S. P. Kouzaris, Physica D 195 (2004) 50.
- [3] S. D. Zhu, Discrete Volterra equation via Exp-function Method, 2007 International Symposium on Nonlinear Dynamics 27-30 Oct., 2007, Shanghai, China.
- [4] S. Y. Lou, B. Tong, H. C. Hu and X. Y. Tang,
- [5] V. A. Brazhnyi and V. V. Konotop, Phys. Rev. E 72 (2005) 026616.
- [6] J. A. Gear, Stud. Appl. Math. 72 (1985) 95.
- [7] H. C. Hu, B. Tong and S. Y. Lou, Phys. Lett. A 351 (2006) 403.
- [8] S. V. Maanakov, Sov. Phys. JETP 40 (1975) 269.
- [9] M. J. Ablowitz, SIAM Review 19 (1977) 663.
- [10] M. J. Ablowitz and J. F. Ladik, Stud. Appl. Math. 57 (1977) 1.
- [11] W. X. Ma, Phys. Lett. A 301 (2002) 35;
W. X. Ma and K. Maruno, Physica A 343 (2004) 219.

